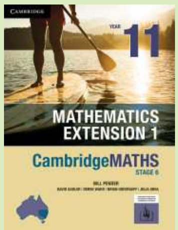
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1
YEAR 11 COURSE

Topic summary and exercises:

With references to



(X1) Further Work with Functions

Name: ....

Initial version by H. Lam, 2019. Last updated February 19, 2024.
Based on the work from the legacy syllabuses by R. Trenwith, 1995–2010, subsequently maintained by H. Lam, 2011-8.
Various corrections by students & members of the Department of Mathematics at North Sydney Boys and Normanhurst Boys High Schools.

The questions and structure of document of the *Graphical Techniques* section arise from various sources, including

- *Graphs* Books 1 & 2, P. Marsh.
- *Sketching Graphs, Extension 2*, R. Trenwith.
- Assistance from M. Ziaziaris.
- GeoGebra animations by E. Chiem (Baulkham Hills High School)

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

Beware! Heed warning.

Mathematics Advanced content.

Mathematics Extension 1 content.

Literacy: note new word/phrase.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME11-1 uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses

ME11-2 manipulates algebraic expressions and graphical functions to solve problems

Syllabus subtopics

ME-F1 Further Work with Functions

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *Cambridge Year 11 3 Unit* (Pender, Sadler, Shea, & Ward, 1999), *Cambridge Year 12 3 Unit* (Pender, Sadler, Shea, & Ward, 2000) and *Sydney Grammar 4 Unit notes* (Sadler & Ward, 2014) will be completed at the discretion of your teacher.
- Unless specified, references to exercises are from Pender et al. (1999).
- Remember to copy the question into your exercise book!

Contents

Part I

Inequalities

Section 1

Linear inequalities

Important note

- The sign of the inequality will .... when .... or .... by a .... number.
- Split any double-ended inequalities into two separate ones.

Example 1

Solve $20 > 2 - 3x \geq 8$, sketching the solution on a number line. **Answer:** $x \in (-6, -2]$

Further exercises

Ex 5A

- Q2-3

Section 2

Quadratic and polynomial inequalities

Learning Goal(s)

Knowledge
Solving polynomial inequalities

Skills
Determine the range of values satisfying an inequality

Understanding
The need to examine the graph to efficiently solve inequalities

By the end of this section am I able to:

3.1 Solve quadratic inequalities using both algebraic and graphical techniques.

2.1 Quadratic

Important note

- Draw picture! Solving it purely algebraically will definitely cause problems.
- A polynomial inequality should be solved by .... the polynomial and solved ....

Example 2

Solve:

(a) $x^2 > 9$ (b) $x + 6 \geq x^2$

Answer: (a) $x \in) - 3, 3($ (b) $x \in [-2, 3]$

2.2 Polynomial

Example 3

Solve $(x+2)(x+1)(x-3) < 0$.

Exercises

Solve using graphical means:

1. $(x-2)(x+3) < 0$

2. $(x-2)(x+3) > 0$

3. $x^2+5x+6 \leq 0$

4. $x^2+5x+6 > 0$

5. $x(x+1)(x-1) \leq 0$

6. $-x(x+1)(x-1) \leq 0$

7. $x^3-5x^2+4x \geq 0$

8. $x(x-2)^2 > 0$

9. $x(x-2)^2 \leq 0$

Answers

1. $-3 < x < 2$ **2.** $x < -3,\ x > 2$ **3.** $-3 \leq x \leq -2$ **4.** $x < -3,\ x > -2$ **5.** $x \leq -1,\ 0 \leq x \leq 1$ **6.** $-1 \leq x \leq 0,\ x \geq 1$ **7.** $0 \leq x \leq 1,\ x \geq 4$ **8.** $x > 0,\ x \neq 2$ **9.** $x \leq 0,\ x = 2$

Further exercises

Ex 5A

- Q4-6, 9, 13, 16

Ex 5B

- Q1, 4-5, 8

Section 3

Absolute value inequalities

Learning Goal(s)

Knowledge
Solving inequalities involving absolute values

Skills
Using accurate graphing techniques to solve inequalities

Understanding
Recognise the complications in algebraic methods

By the end of this section am I able to:

3.2 Solve absolute value inequalities.

3.3 Solve absolute value equations and inequations algebraically.

3.1 Harder absolute value equations

Important note

Draw picture! Solving it purely algebraically may cause problems.

3.1.1 Absolute value with additional variables

Example 4

Solve $|2x-1|=3$. **Answer:** $x=-1,2$

Example 5

Solve $|3x - 7| = |2x - 3|$.

Important note

Draw picture to verify the number of solutions.

Example 6

Find t if $t + |t| = 8$

Example 7

Find a if $|a - 3| + a + 3 = 0$

3.1.2 Absolute value by cases

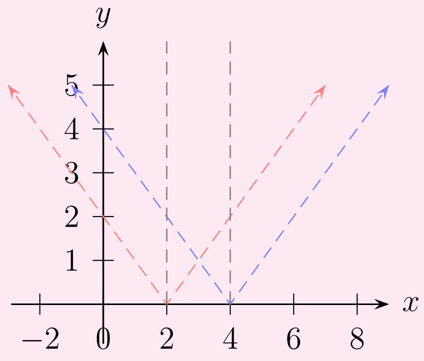
Example 8

(a) ⚠ Sketch $y = |x-2| + |x-4|$.

(b) Hence or otherwise, solve $|x-2| + |x-4| = 4$.

Steps

1. Sketch $y = |x-2|$ and $y = |x-4|$ on the same set of axes:



2. Perform addition of ordinates and take branches to sketch $y = |x-2| + |x-4|$:

- $x < 2$
- $2 < x < 4$
- $x > 4$

(Draw on the same set of axes as previous part)

3. Examine the three 'branches' of $y = |x-2| + |x-4|$:

- $x < 2$:
  - .... branch $y = |x-2|$
  - .... branch $y = |x-4|$

  $\therefore y =$ ....
- $2 < x < 4$:
  - .... branch $y = |x-2|$
  - .... branch $y = |x-4|$

  $\therefore y = \ldots$
- $x > 4$:
  - .... branch $y = |x-2|$
  - .... branch $y = |x-4|$

  $\therefore y =$ ....

4. Solve $\begin{cases} y = |x-2| + |x-4| \\ y = 4 \end{cases}$ simultaneously by sketching $y = 4$, and examining intersection with appropriate branches.

Example 9

(a) Sketch $y = |x+1| + |x-2|$.

(b) Hence or otherwise, solve $|x+1| + |x-2| = 6$.

Example 10

Solve $|x-3|+|x+1|=4$.

Important note

- What does Example 10 highlight about making an accurate sketch and understanding the graphs?

3.2 Solution of inequalities

Example 11

[2012 2U Q11] (2 marks) Solve $|3x - 1| < 2$.

Important note

Which two graphs should be sketched and their points of intersection found in order to verify the solution?

Example 12

(a) Solve $|2x - 5| = x + 2$. **Answer:** $x = 1, 7$

(b) Hence solve $|2x - 5| \geq x + 2$.

Example 13

[2016 NBHS Ext 1 Assessment Task 1] ⚠

(i) On the same set of axes, sketch the graphs of $y = 2|x|$ and $y = |x+3|$. **4**

(ii) Give the coordinates of the point(s) of intersection of the two graphs. **2**

(iii) Hence solve $2|x| > |x+3|$. **1**

Example 14

Sketch $y = \dfrac{|x-1|}{(x-1)(x+3)}$.

Example 15

[2013 Ext 1 Q10] ⚠ Which inequality has the same solutions as

$$|x+2|+|x-3|=5$$

(A) $\dfrac{5}{3-x}\geq 1$

(B) $\dfrac{1}{x-3}-\dfrac{1}{x+2}\leq 0$

(C) $x^2-x-6\leq 0$

(D) $|2x-1|\geq 5$

Example 16

[2016 NBHS Ext 1 Assessment Task 1] Solve for x: $|x^2-5|=5x+9$

Exercises

1. Find all solutions to the following:

(a) $|3x-7| = |2x-3|$ (e) $|x+1| = |2x+7|$ (i) $|2x+1| = |x-2|$
(b) $|7-2x| = |x-2|$ (f) $|5-3x| = |x+3|$ (j) $2\,|x+8| = 3\,|x+5|$
(c) $|2+x| = |-x|$ (g) $|7x-4| = |3x+16|$ (k) $5\,|x-7| = |9x+1|$
(d) $|3x-1| = |5+2x|$ (h) $|9x+2| = |3x-4|$ (l) $|7x-3| = 4\,|x+6|$

2. Solve each of the following and verify that the solutions are valid:

(a) $|2x-1| = x+7$ (e) $|5-3x| = x+1$ (i) $|2x| = 9-x$
(b) $|x+7| = 2x-1$ (f) $5-3x = |x+1|$ (j) $|2x+5| = 3x+9$
(c) $|2x-11| = 3x-4$ (g) $|3x+1| = 2x+4$ (k) $|6x-5| = 5x+27$
(d) $2x-11 = |3x-4|$ (h) $|4x-1| = 2x+7$ (l) $|4-2x| = x-2$

3. Solve the following inequalities and graph each solution on the number line.

(a) $|8x| < 24$
(b) $|x+6| < 8$
(c) $|2x-1| < 5$
(d) $|x| > 3$
(e) $|x+4| > 7$
(f) $|3x-2| > 10$
(g) $|4x+3| > 19$
(h) $|5x-2| < 3$
(i) $|2x+5| < 13$
(j) $|2x+6| \leq 12$
(k) $|7x-3| \geq 4$
(l) $|6x+5| < 1$
(m) $\left|\frac{x}{2}\right| < 4$
(n) $\left|\frac{x}{3}\right| > 5$
(o) $\left|\frac{x-1}{2}\right| < 3$
(p) $\left|\frac{x+1}{4}\right| > 1$
(q) $|2x+5| < 6$
(r) $|9x-4| \leq 5$
(s) $|5x-1| > 1$
(t) $\left|\frac{2x}{5}\right| \leq 4$
(u) $\left|\frac{3x-1}{2}\right| > 1$
(v) $\left|\frac{2x+5}{3}\right| < 2$

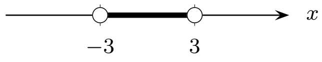
4. ⚠ Solve by drawing appropriate sketches and examining branches:

(a) $|x| + |x-4| = 2$ (c) $|x-3| + |x+1| = 2$
(b) $|x-3| + |x+1| = 5$ (d) $|x-1| + |x+3| = 4$

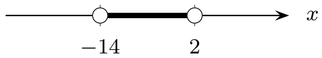
Answers

1. (a) $x = 2, 4$ (b) $x = 3, 5$ (c) $x = -1$ (d) $x = 6, -\frac{4}{5}$ (e) $x = -6, -\frac{8}{3}$ (f) $x = 4, \frac{1}{2}$ (g) $x = 5, -\frac{6}{5}$ (h) $x = -1, \frac{1}{6}$ (i) $x = \frac{1}{3}, -3$ (j) $x = 1, -\frac{31}{5}$ (k) $x = -9, \frac{17}{7}$ (l) $x = 9, -\frac{21}{11}$ **2.** (a) $x = 8, -2$ (b) $x = 8$ (c) $x = 3$ (d) no solutions (e) $x = 1, 3$ (f) $x = 1$ (g) $x = -1, 3$ (h) $x = -1, 4$ (i) $x = 3, -9$ (j) $x = -\frac{14}{5}$ (k) $x = -2, 32$ (l) $x = 2$

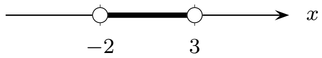
4. (a) $-3 < x < 3$



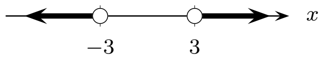
(b) $-14 < x < 2$



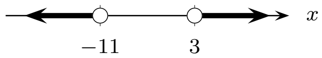
(c) $-2 < x < 3$



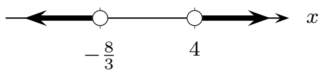
(d) $x > 3$ or $x < -3$



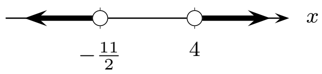
(e) $x > 3$ or $x < -11$



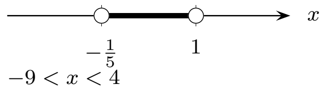
(f) $x > 4$ or $x < -\frac{8}{3}$



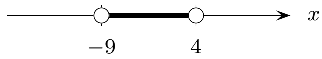
(g) $x > 4$ or $x < -\frac{11}{2}$



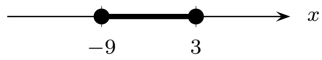
(h) $-\frac{1}{5} < x < 1$



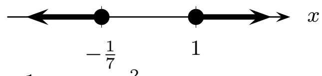
(i) $-9 < x < 4$



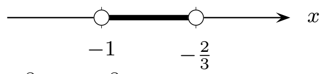
(j) $-9 \leq x \leq 3$



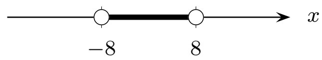
(k) $x \geq 1$ or $x \leq -\frac{1}{7}$



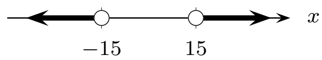
(l) $-1 < x < -\frac{2}{3}$



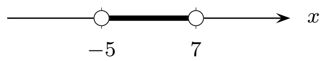
(m) $-8 < x < 8$



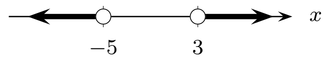
(n) $x > 15$ or $x < -15$



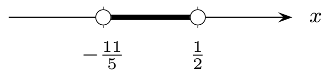
(o) $-5 < x < 7$



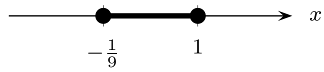
(p) $x > 3$ or $x < -5$



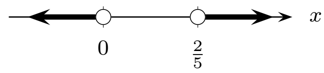
(q) $-\frac{11}{5} < x < \frac{1}{2}$



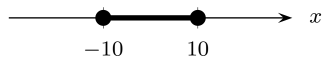
(r) $-\frac{1}{9} \leq x \leq 1$



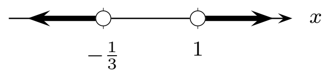
(s) $x > \frac{2}{5}$ or $x < 0$



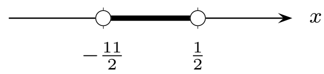
(t) $-10 \leq x \leq 10$



(u) $x > 1$ or $x < -\frac{1}{3}$



(v) $-\frac{11}{2} < x < \frac{1}{2}$



5. (a) No solution (b) $x = -\frac{3}{2}, \frac{7}{2}$ (c) No solution (d) $-3 \leq x \leq 1$

Further exercises

Ex 5A

- Q12, 15, 17, 18

Section 4

Rational expression inequalities

Learning Goal(s)

Knowledge
Solving rational expression inequalities

Skills
Identifying when to multiply through by the square of the denominator in inequalities

Understanding
The uncertainty involved in inequalities with unknowns in the denominator

By the end of this section am I able to:

3.4 Solve inequalities involving rational expressions, including those with the unknown in the denominator.

Laws/Results

An inequality with an unknown in the denominator will make it uncertain where the function becomes positive or negative.

To avoid this problem,

- multiply by the .... of the denominator, or,
- .... and solve graphically.

4.1 Expressions resulting in a quadratic inequality

Gentle reminder

What is you favourite f-word? ☺

....

Example 17

Solve $\dfrac{5}{x-4} \geq 1$

Graphical

Algebraic

Important note

Remember to exclude a particular value of x!

Example 18

For what values of x is $\dfrac{2x+3}{x-4} > 1$?

Answer: $x < -7$ or $x > 4$.

Graphical

Algebraic

Important note

⚠ Remember to exclude a particular value of x!

4.2 Expressions resulting in a polynomial inequality

Example 19

Solve $\dfrac{6}{x} \geq x - 5$.

Answer: $0 < x \leq 6$ or $x \leq -1$.

Graphical

Algebraic

Important note

Remember to exclude a particular value of x!

Example 20

[2002 NEAP Ext 1 Q3] (3 marks) Solve the inequality $\dfrac{x}{x^2-4} \leq 0$ for x.

Answer: $x \in \{(-\infty, -2) \cup [0, 2)\}$.

Important note

Remember to exclude particular value(s) of x!

Exercises

Source Portions taken from Fitzpatrick (1984, Ex 23(a))

1. $x^2 - 2x - 15 \le 0$

2. $x(x-1) \le 6$

3. $4x^2 - 12x + 10 > 0$

4. $x^2 + 4x + 13 \le 0$

5. $-3x^2 + 10x + 8 \le 0$

6. $2x^2 + 5x + 2 \ge 0$

7. $(x-1)(x+3)(x-2) < 0$

8. $(2+x)(x-5)(x+1) > 0$

9. $x^2(x-1) \le 0$

10. $\dfrac{x-3}{x+1} > 0$

11. $\dfrac{1}{x} > 6$

12. $\dfrac{x-2}{x+3} > -2$

13. $\dfrac{4x-3}{2x+1} \le 3$

14. $\dfrac{1}{2x-1} \le 2$

15. $\dfrac{2}{1-x} > -1$

16. $\dfrac{1}{x-2} > 1$

17. $\dfrac{2}{2-x} \ge 3$

18. $\dfrac{1}{x} < \dfrac{1}{4}$

19. $\dfrac{1}{x-3} > 2$

20. $\dfrac{8}{x+1} > 2$

21. $\dfrac{4}{2x-1} < \dfrac{2}{3}$

22. $\dfrac{12}{3x+2} > 4$

23. $\dfrac{6}{5x-2} < 2$

24. $\dfrac{1}{(x-1)(x-3)} \le -1$

25. $\dfrac{7}{(3-x)(x+3)} > -1$

26. $3x^2 + 5x + 1 > \dfrac{1}{4}$

27. $\dfrac{2x-4}{x+3} > \dfrac{x+2}{2x+6}$

28. $\dfrac{1}{5x^2 - 2x - 7} < \dfrac{5}{13}$

29. $2^{2x} - 5\left(2^x\right) + 4 \le 0$

30. $2^{2x} - 2\left(2^x\right) \le -1$

31. ⚠ Find values of x for which the following inequations are simultaneously satisfied:

(a) $\dfrac{x+4}{x+6} < 0$ and $\dfrac{x-6}{x-4} > 1$

(b) $x + \dfrac{1}{|x|} > 0$ and $x^2 - x - 2 < 0$

(c) $x^2 - 5x + 4 \le 0$ and $6 - x - x^2 > 0$

Answers

1. $-3 \le x \le 5$ **2.** $-2 \le x \le 3$ **3.** $x \in \mathbb{R}$ **4.** no solution **5.** $x \le -\frac{2}{3}$ or $x \ge 4$ **6.** $x \le -2$ or $x \ge -\frac{1}{2}$ **7.** $x < -3$ or $1 < x < 2$ **8.** $-2 < x < -1$ or $x > 5$ **9.** $x \le 1$ **10.** $x < -1$ or $x > 3$ **11.** $0 < x < \frac{1}{6}$ **12.** $x < -3$ or $x > -\frac{4}{3}$ **13.** $x \le -3$ or $x > -\frac{1}{2}$ **14.** $x < \frac{1}{2}$ or $x \ge -\frac{4}{3}$ **15.** $x < 1$ or $x > 3$ **16.** $2 < x < 3$ **17.** $\frac{4}{3} \le x < 2$ **18.** $x < 0$ or $x > 4$ **19.** $3 < x < \frac{7}{2}$ **20.** $-1 < x < 3$ **21.** $x < \frac{1}{2}$ or $x > \frac{7}{2}$ **22.** $-\frac{2}{3} < x < \frac{1}{3}$ **23.** $x < \frac{2}{5}$ or $x > 1$ **24.** $1 < x < 3$ **25.** $x < -4$ or $-3 < x < 3$ or $x > 4$ **26.** $x > -\frac{1}{6}$ or $x < -\frac{3}{2}$ **27.** $x < -3$ or $x > \frac{10}{3}$ **28.** $x < -\frac{6}{5}$ or $-1 < x < \frac{7}{5}$ or $x > \frac{8}{5}$ **29.** $0 \le x \le 2$ **30.** $x = 0$ **31.** (a) $-6 < x < -4$ (b) $-1 \le x < 2$ (c) $-1 < x < 0$ or $0 < x < 2$

Further exercises

Ex 5A

- Q11, 14

Ex 5B

- Q7, 10

Part II

Graphical relationships

Section 5

Reflection about coordinate axes

Learning Goal(s)

Knowledge
Curve sketching techniques

Skills
Applying a curve sketching menu to graph functions

Understanding
The reflection of the graph $y = f(x)$ for the graphs $y = |f(x)|$ and $y = f(|x|)$

By the end of this section am I able to:

3.5 Develop an introductory set of techniques a curve sketching menu to be able to sketch graphs of functions.

3.6 Develop a more complete curve sketching menu.

3.7 Examine the relationship between the graph of $y = f(x)$ and the graphs of $y = |f(x)|$ and $y = f(|x|)$ and hence sketch the graphs.

5.1 Basic curve sketching techniques

Steps

1. Determine the ....

2. Look for symmetry (........... , even or ....).

3. Locate x and y ....

4. Determine where the graph is .... and where it is ....

5. Consider .... and .... asymptotes. Also consider the .... behaviour as $x \to \pm\infty$.

5.2 Reflection properties

- $f(-x)$: reflection about .... axis.
- $-f(x)$: reflection about .... axis.
- $|f(x)|$: reflect .... about .... axis.
  - Use basic definition of absolute value:

$$|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

- $f(|x|)$:
  - Draw sketch of $y = f(x)$ for $x \geq 0$.
  - Reflect about y axis.
  - Use basic definition of absolute value:

$$f(|x|) = \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0 \end{cases}$$

- ⚠ $|y| = f(x)$: reflect positive values of $f(x)$ along x axis.
  - $f(x) > 0$, reflect along y axis ($\pm y$ exists)
  - $f(x) < 0$, not defined.

Important note

Not explicitly stated in the syllabus, but included from the legacy '4 Unit' course for completeness.

5.3 Odd/Even functions

Definition 1

- An *odd* function has .... about the .... Algebraically,

  ....

- An *even* function has .... about the y axis. Algebraically,

  ....

Example 21

Sketch the graph of $y = \sqrt{-x}$.

Example 22

Sketch the graph of $y = |x^2 - 2x|$.

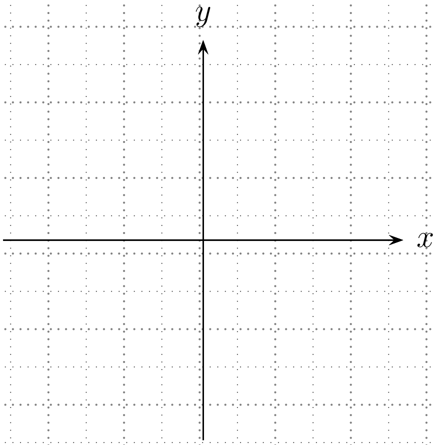
Example 23

For $y = f(x) = (x-2)(x-4)$, sketch:

1. $f(x)$ **3.** $-f(x)$ **5.** $f(|x|)$

2. $f(-x)$ **4.** $|f(x)|$ **6.** $|y| = f(x)$.

1.



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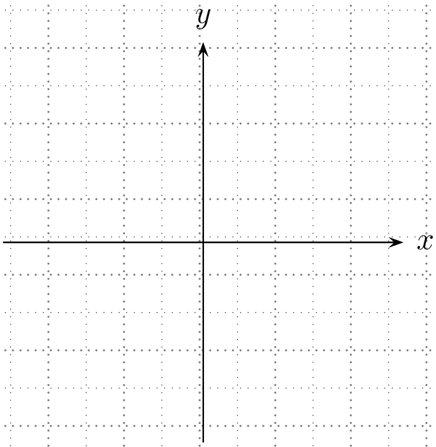

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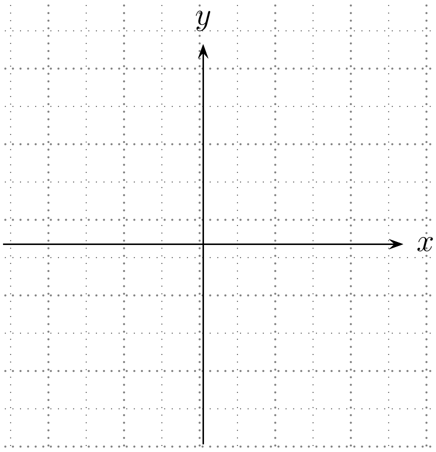

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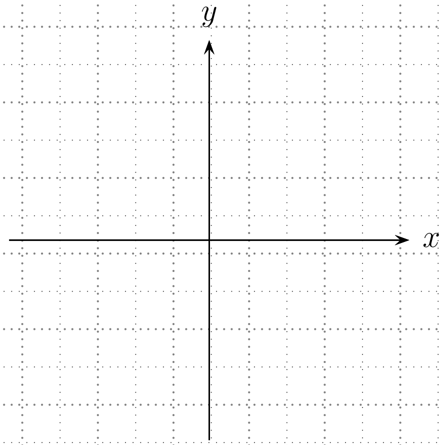

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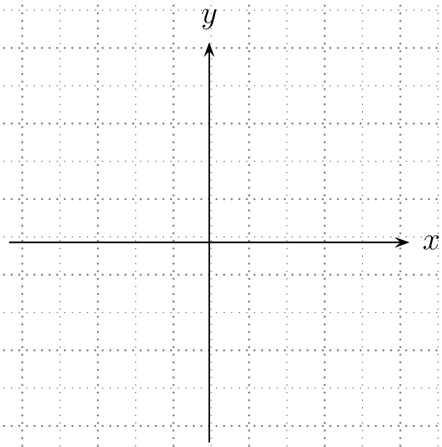

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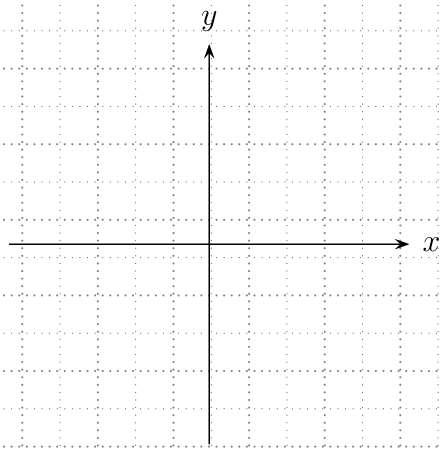


Exercises

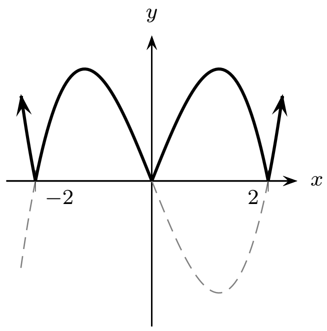
Sketch the following graphs:

1. $y = \dfrac{1}{3^x} - 1$
2. $y = |x(x^2 - 4)|$
3. $y = 5|x| - 2$
4. $y = |x^2| - |x|$
5. $|y| = |x^2 - 1|$
6. $|x| + |y| = 1$
7. $|x + y| = 1$

Answers

1. $y = \frac{1}{3^x} - 1$: GeoGebra

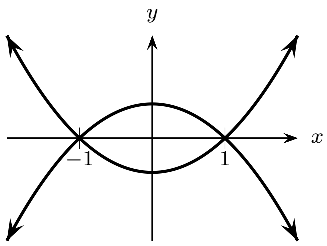
2. $y = |x(x^2 - 4)|$



3. $y = 5|x| - 2$: GeoGebra

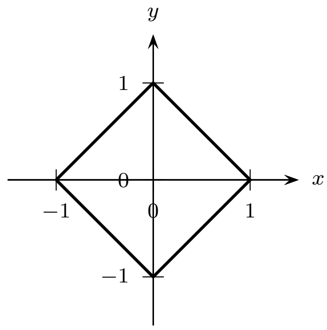
4. $y = |x^2| - |x|$
   - $x \geq 0$
   $$y = x^2 - x$$
   - $x < 0$,
   $$y = x^2 - (-x)$$

5. $|y| = |x^2 - 1|$

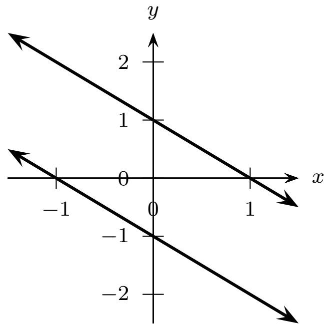


6. $|x| + |y| = 1$
   Consider cases:
   - $y = 1 - |x|$
   - $-y = 1 - |x|$

   Also, $x, y < 1$ for this to be true.



7. $|x + y| = 1$



Further exercises

Ex 4B
- Q5-6

Ex 5E
- Q2, 5-6, 8
- Q10, 14

Section 6

Addition/subtraction of ordinates: $y = f(x) \pm g(x)$

Learning Goal(s)

Knowledge
Sketching the sum and difference of two functions

Skills
Adding and subtracting the ordinates of two functions

Understanding
The characteristics of sums and differences of two functions, including oblique asymptotes

By the end of this section am I able to:

3.8 Examine the relationship between the graphs of $y = f(x)$ and $y = g(x)$ and the graphs of $y = f(x) + g(x)$ and $y = f(x)g(x)$ and hence sketch the graphs.

Steps

1. Ensure that the equation is written in the form $y = f(x) + g(x)$.
   - Test for even/odd function.
2. Find x and y intercepts for $y = f(x) + g(x)$.
3. Draw dotted sketches of $y = f(x)$ and $y = g(x)$ on the same set of axes.
4. Draw perpendicular lines to the x axis that cut $y = f(x)$ and $y = g(x)$.
5. Join points to represent graph of $y = f(x) + g(x)$.

GeoGebra

`Adding-trig-curves.ggb`

Example 24

Sketch $y = x + \dfrac{1}{x}$.

Example 25

Sketch $y = x - \dfrac{1}{x}$.

Example 26

Sketch $y = \sqrt{x-1} + \sqrt{5-x}$.

Exercises

Sketch the following graphs.

1. $y = x + \dfrac{1}{x^2}$

2. $y = x - \dfrac{1}{x^2}$

3. $y = \dfrac{e^x + e^{-x}}{2}$ $(e \approx 2.71828\cdots)$

4. $y = \dfrac{e^x - e^{-x}}{2}$

5. $y = \dfrac{1}{x + e^x}$

6. $y = \dfrac{1}{x^2 - \frac{1}{x}}$

7. (a) Show that

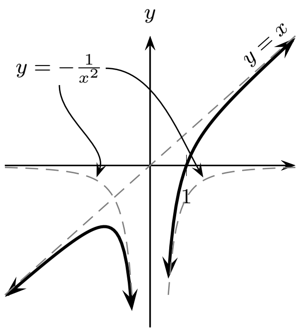
$$\frac{x^3 - x^2 + 1}{x - 1} = x^2 + \frac{1}{x - 1}$$

(b) Hence sketch $y = \dfrac{x^3 - x^2 + 1}{x - 1}$.

Answers

1. $y = x + \dfrac{1}{x^2}$: GeoGebra

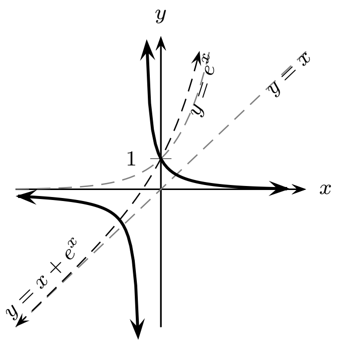
2. $y = x - \dfrac{1}{x^2}$



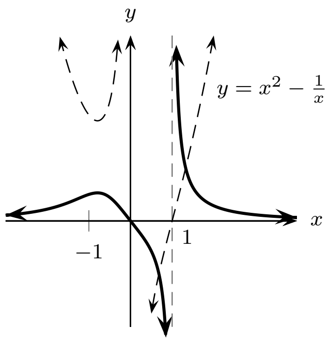
3. $y = \dfrac{e^x + e^{-x}}{2}$: GeoGebra

4. $y = \dfrac{e^x - e^{-x}}{2}$: GeoGebra

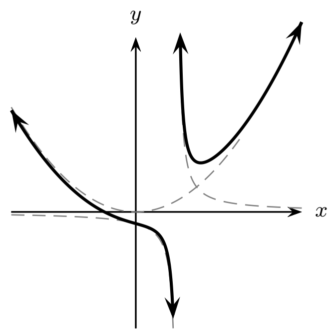
5. $y = \dfrac{1}{x + e^x}$



6. $y = \dfrac{1}{x^2 - \frac{1}{x}}$



7. $y = \dfrac{x^3 - x^2 + 1}{x - 1}$



Further exercises

Ex 5D

- Q1-2, 4-5, 7-8, 10, 14-15

Section 7

Multiplication of ordinates

Learning Goal(s)

Knowledge
Sketching the product of two functions

Skills
Multiplying the ordinates of two functions

Understanding
The characteristics of products of two functions

By the end of this section am I able to:

3.8 Examine the relationship between the graphs of $y = f(x)$ and $y = g(x)$ and the graphs of $y = f(x) + g(x)$ and $y = f(x)g(x)$ and hence sketch the graphs.

7.1 Graphs of form $y = xf(x)$

Steps

1. Sketch $y = f(x)$ and $y = x$ (draft)
2. Determine sign of $y = xf(x)$ by considering the sign of $y = x$ and $y = f(x)$ at significant intervals.
3. Dot in lines $x = 1$. Mark the point where this line passes $y = f(x)$ as $y = xf(x)$ passes through this point.
4. Dot in lines $x = -1$. Reflect the point where this line passes $y = f(x)$ in the x axis as $y = xf(x)$ passes through this point.
5. Consider $\lim_{x \to \pm\infty} xf(x)$ to determine possible horizontal asymptotes.
6.
   - For $x \in (-1, 1)$, draw graph shallower than $y = f(x)$.
   - For $x \in (-\infty, -1) \cup (1, \infty)$, draw curve steeper than $y = f(x)$.

Example 27

Sketch the graph of $y = xe^x$. ($e \approx 2.71828 \cdots$)

Exercises

Sketch the following graphs.

1. $y = x\left(x^2 - 9\right)$
2. $y = xe^{-x}$
3. $y = \dfrac{x}{x-2}$
4. $y = \dfrac{x}{x+1}$
5. $y = \dfrac{x}{x^2+1}$
6. $y = x\sqrt{1-x}$
7. $y = x\left(e^x - 1\right)$

7.2 Graphs of the form $y = f(x)g(x)$

Steps

1. Sketch $y = f(x)$ and $y = g(x)$ (draft)
2. Determine whether the function is odd or even, and hence any symmetry.
   - Odd × Odd → Even
   - Odd × Even → Odd
   - Even × Even → Even
3. Consider behaviour of $y = f(x)g(x)$ around the neighbourhood of $x = 0$.
4. Consider $\lim\limits_{x\to\pm\infty} f(x)g(x)$ to determine any asymptotes.
5. Investigate domain of f and g to determine the domain of $y = f(x)g(x)$.
6. Use calculus to determine exact stationary points and points of inflexion.

Example 28

Sketch the graph of $y = x^2e^{-x}$. ($e \approx 2.71828\cdots$)

Further exercises

Ex 5D

- Q3, 6, 9, 11

Section 8

Division of ordinates

Learning Goal(s)

Knowledge
Sketching the quotient of functions

Skills
Dividing the ordinates of functions

Understanding
The characteristics of quotients of functions

By the end of this section am I able to:

3.9 Examine the relationship between the graph of $y = f(x)$ and the graph of $y = \dfrac{1}{f(x)}$ and hence sketch the graphs.

8.1 Graphs of the form $y = \frac{1}{f(x)}$

Steps

1. Draw $f(x)$ (draft)

2. Vertical asymptotes whenever $f(x) = 0$ as $y = \dfrac{1}{f(x)}$ does not exist.

3. Find the y intercept of $y = \dfrac{1}{f(x)}$.

4. Draw lines $y = \pm 1$, as $f(x)$ will pass through these exact same points.

5. Consider magnitude of $y = f(x)$ and hence magnitude of $y = \dfrac{1}{f(x)}$.

- As $x \to \pm\infty$.
- As x approaches asymptotes.

Important note

A maximum value on $f(x)$ will result in a .... value on $\dfrac{1}{f(x)}$, and vice versa.

⚠ Care is needed for graphs such as $y = \tan x$ and $y = \cot x$.

⚠ Later after the *Trigonometry* topic has been taught, when sketching $y = \dfrac{1}{\tan x}$ will take the shape of $y = \cot x$, except when $\tan x = 0$.

Example 29

Sketch $y = \dfrac{1}{(x-1)(x+1)(x-2)}$.

8.2 ⚠ Other manipulations involving $\dfrac{1}{f(x)}$

Important note

Some of the content in this section may not be explicit in the syllabus, but could arise due to the addition/multiplication of ordinates or other techniques previously presented.

Laws/Results

Functions of the form $\dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, follow the asymptotes.

Example 30

Sketch $y = \dfrac{x-1}{x-4}$, showing all important features.

Example 31

⚠ Sketch $y = \dfrac{2x^2}{x^2-9}$, showing all important features.

Example 32

Sketch $y = \dfrac{1}{x-2} + \dfrac{1}{x-8}$

Example 33

Sketch $y = x^2 + \dfrac{1}{x^2}$.

Example 34

[Ex 3G Q17(a)] ⚠ Show that $\dfrac{(x-1)(x+2)}{x-3} = x + 4 + \dfrac{10}{x-3}$, and deduce that $y = \dfrac{(x-1)(x+2)}{x-3}$ has an oblique asymptote $y = x + 4$. Then sketch the graph.

Example 35

[Ex 3G Q17(b)] ⚠ Likewise, sketch $y = \dfrac{x^2-4}{x+1} = x - 1 - \dfrac{3}{x+1}$, showing the oblique asymptote.

Exercises

1. Sketch the following rectangular hyperbolae on separate number planes, stating the domain and range.

(a) $y = \dfrac{1}{x}$

(b) $y = \dfrac{3}{x-3}$

(c) $y = \dfrac{-3}{x-3}$

(d) $y = 1 + \dfrac{1}{x+2}$

(e) $y = 4 - \dfrac{3}{x-3}$

(f) $y = \dfrac{2}{x-4} + 1$

(g) $y = -2 - \dfrac{1}{x+1}$

(h) $y = \dfrac{1}{2x-3} - 3$

(i) $y = \dfrac{x+2}{x+1}$

(j) $y = \dfrac{x}{x+1}$

(k) $y = \dfrac{2x+1}{x-2}$

(l) $y = \dfrac{3x-2}{x+3}$

(m) $y = \dfrac{1-x}{1+x}$

2. Reciprocal functions

(a) i. On the same set of axes, sketch the graphs of $y = x$ and $y = -\dfrac{1}{x}$.

ii. By addition of ordinates, sketch the graph of $y = x - \dfrac{1}{x}$.

(b) Sketch the following by the addition of ordinates:

i. $y = 2x + \dfrac{1}{x}$

ii. $y = \dfrac{3}{x} - 2x$

iii. $y = x^2 + \dfrac{1}{x}$

iv. $y = x^2 - \dfrac{1}{x}$

(c) Sketch the following graphs of reciprocal functions, stating the domain and range:

i. $y = \dfrac{1}{x^2}$

ii. $y = \dfrac{1}{(x-2)^2}$

iii. $y = \dfrac{1}{x^3 - x}$

iv. $y = \dfrac{2}{x^2+1}$

v. $y = \dfrac{2}{x^2-1}$

vi. $y = \dfrac{1}{\sqrt{4-x^2}}$

vii. $y = \dfrac{2}{(x-3)(x+2)}$

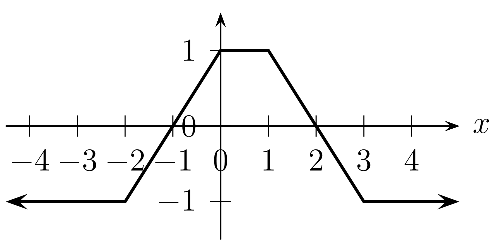
viii. $y = \dfrac{1}{x^2+x}$

ix. $y = \dfrac{x+1}{(x-1)(x+2)}$

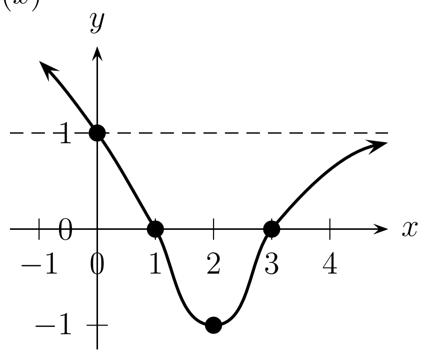
x. $y = \dfrac{x^2-4}{x^2+2x-3}$

3. $f(x)$ is shown in the following diagrams. Sketch $\dfrac{1}{f(x)}$.

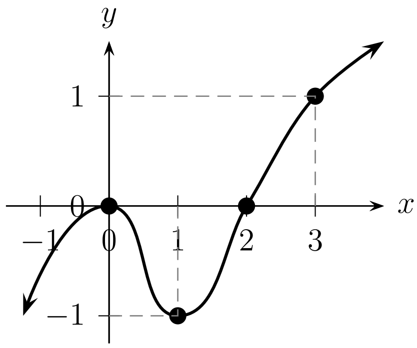
(i)



(iii)



(ii)



4. Make neat sketches of the following graphs:

(a) $y = 2 - x$ and $y = \dfrac{1}{2-x}$

(b) $y = 3x + 1$ and $y = \dfrac{1}{3x+1}$

(c) $y = x^2 - 4$ and $y = \dfrac{1}{x^2-4}$

(d) $y = 9 - x^2$ and $y = \dfrac{1}{9-x^2}$

(e) $y = x^2 - x - 2$ and $y = \dfrac{1}{x^2-x-2}$

(f) $y = \sqrt{x}$ and $y = \dfrac{1}{\sqrt{x}}$

(g) $y = |x|$ and $y = \dfrac{1}{|x|}$

(h) $y = 2^x$ and $y = \dfrac{1}{2^x}$

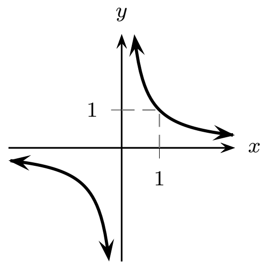
(i) $y = \sin x$ and $y = \dfrac{1}{\sin x}$

(j) $y = (x-1)\left(x^2 - 3x - 4\right)$ and $y = \dfrac{1}{(x-1)\left(x^2-3x-4\right)}$

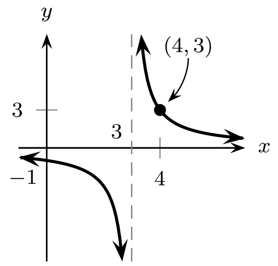
(k) $y = |2x+1|$ and $y = \dfrac{1}{|2x+1|}$

Answers

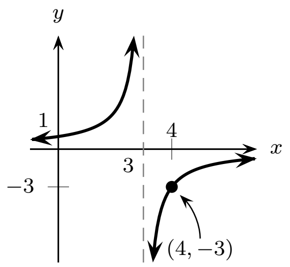
1. (a) $D = \{x : x \neq 0\}$
$R = \{y : y \neq 0\}$



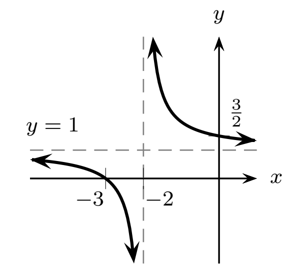
(b) $D = \{x : x \neq 3\}$
$R = \{y : y \neq 0\}$



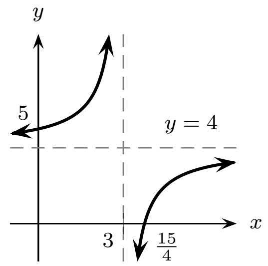
(c) $D = \{x : x \neq 3\}$
$R = \{y : y \neq 0\}$



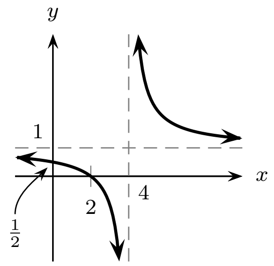
(d) $D = \{x : x \neq -2\}$
$R = \{y : y \neq 1\}$



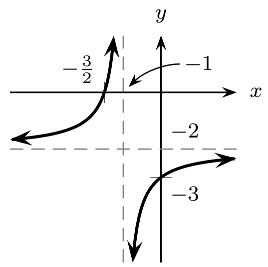
(e) $D = \{x : x \neq 3\}$
$R = \{y : y \neq 4\}$



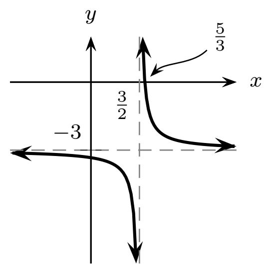
(f) $D = \{x : x \neq 4\}$
$R = \{y : y \neq 1\}$



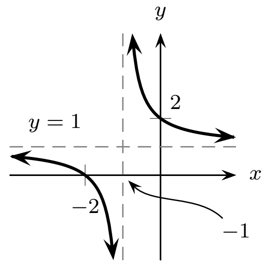
(g) $D = \{x : x \neq -1\}$
$R = \{y : y \neq -2\}$



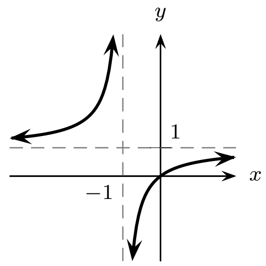
(h) $D = \left\{x : x \neq \frac{3}{2}\right\}$
$R = \{y : y \neq -3\}$



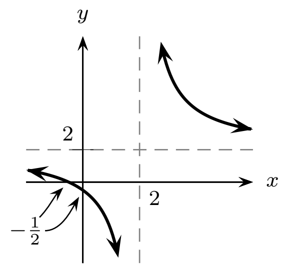
(i) $D = \{x : x \neq -1\}$
$R = \{y : y \neq 1\}$



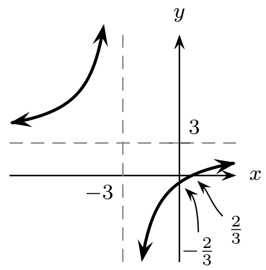
(j) $D = \{x : x \neq -1\}$
$R = \{y : y \neq 1\}$



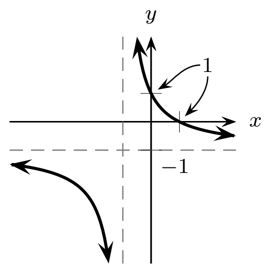
(k) $D = \{x : x \neq 2\}$
$R = \{y : y \neq 2\}$



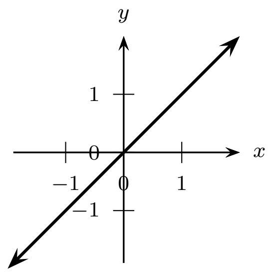
(l) $D = \{x : x \neq -3\}$
$R = \{y : y \neq 3\}$



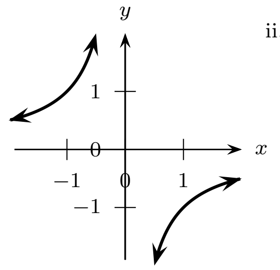
(m) $D = \{x : x \neq -1\}$
$R = \{y : y \neq -1\}$



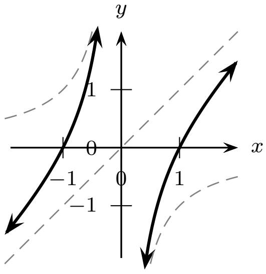
2. (a)



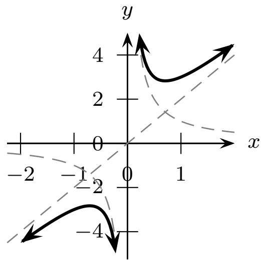
i.



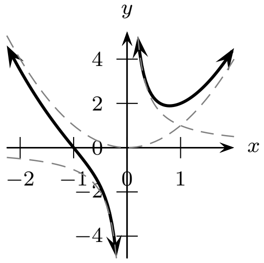
ii.



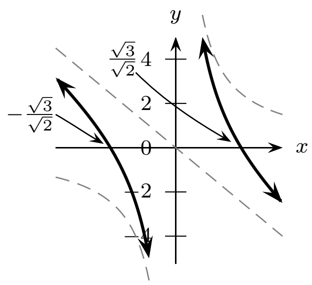
(b) i. $y = 2x + \dfrac{1}{x}$



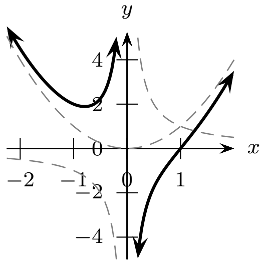
iii. $y = x^2 + \dfrac{1}{x}$



ii. $y = \dfrac{3}{x} - 2x$



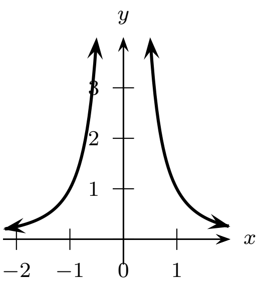
iv. $y = x^2 - \dfrac{1}{x}$

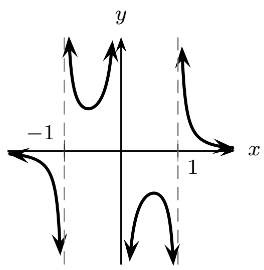


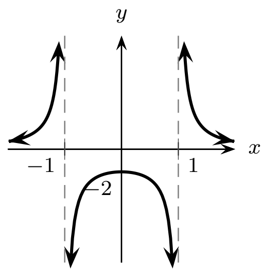
(c) i. $D = \{x : x \neq 0\}$
$R = \{y : y > 0\}$

iii. $D = \{x : x \neq 0, \pm 1\}$
$R = \{y : y \neq 0\}$

v. $D = \{x : x \neq \pm 1\}$
$R = \{y : y \leq -2, y > 0\}$



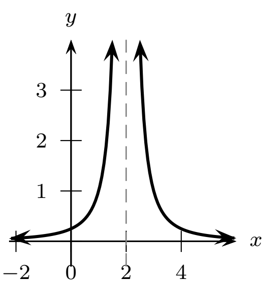


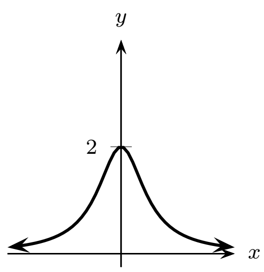


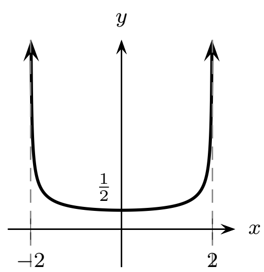
ii. $D = \{x : x \neq 2\}$
$R = \{y : y > 0\}$

iv. $D = \{x : x \in \mathbb{R}\}$
$R = \{y : y > 0\}$

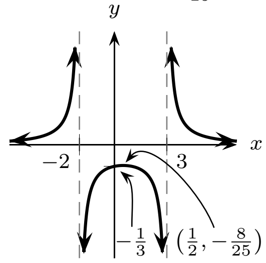
vi. $D = \{x : -2 < x < 2\}$
$R = \left\{y : y > \frac{1}{2}\right\}$



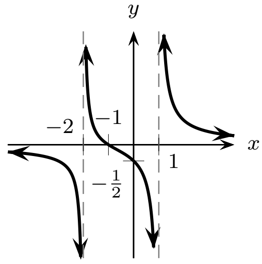




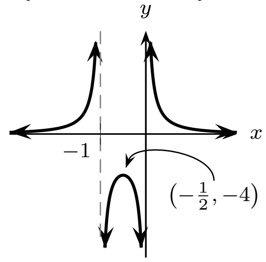
vii. $D = \{x : x \neq 3, -2\}$
$R = \{y : y > 0, y \leq -\frac{8}{25}\}$



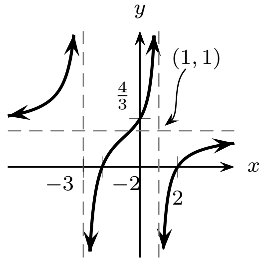
ix. $D = \{x : x \neq 1, -2\}$
$R = \{y : y \in \mathbb{R}\}$



viii. $D = \{x : x \neq -1, 0\}$
$R = \{y : y \leq -4, y > 0\}$



x. $D = \{x : x \neq -3, 1\}$
$R = \{y : y \in \mathbb{R}\}$



3. Remaining exercises: check GeoGebra where possible.

Further exercises

Ex 5C

- Q5-12, 14

Section 9

Square root graphs: $y = \pm\sqrt{f(x)}$

Learning Goal(s)

Knowledge
Sketching the square root of functions

Skills
Applying a sequence of strategies to sketch square root graphs

Understanding
The characteristics of different square roots of functions

By the end of this section am I able to:

3.10 Examine the relationship between the graph of y = f(x) and the graphs of $y^2 = f(x)$ and $y = \sqrt{f(x)}$ and hence sketch the graphs.

3.11 Apply knowledge of graphical relationships to solve problems in practical and abstract contexts.

3.12 Solve a range of equations and inequations using graphing techniques.

Steps

1. Draw rough sketch of $y = f(x)$

2. Erase $f(x)$ where $f(x) < 0$

3. Reflect section of $f(x)$ about x axis to obtain $-f(x)$,

4. Smooth out and "flatten" to obtain $\pm\sqrt{f(x)}$

Note that if $f(x) = (x-a)^r g(x)$, and if

- $r = 1$, then $y = \pm\sqrt{f(x)}$ will behave like a .... near the root.
- $r = 2$, then $y = \pm\sqrt{f(x)}$ will behave like a .... near the root.
- $r = 3$, then $y = \pm\sqrt{f(x)}$ will behave like a .... near the root.
- $r = 4$, then $y = \pm\sqrt{f(x)}$ will behave like a .... near the root.

Example 36

Sketch $y = \pm\sqrt{x^2 - 2x}$.

Example 37

Sketch $y = \pm\sqrt{(x+2)(x-1)(x-3)}$.

Example 38

Sketch $y = \pm\sqrt{(x+1)^2(2-x)}$.

Example 39

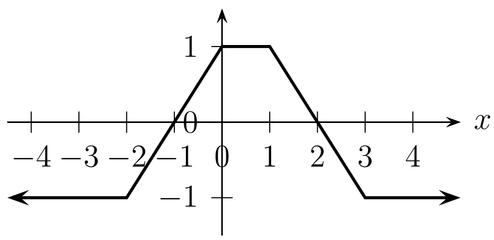
Sketch $y = \pm\sqrt{(x+1)^2x^2(x-1)}$.

Exercises

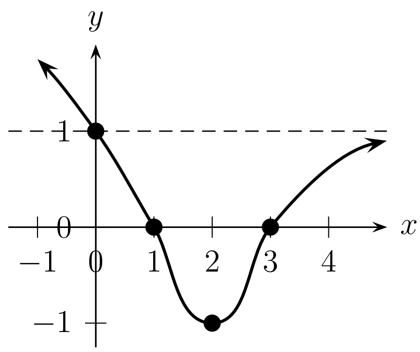
1. $f(x)$ is shown in the following diagrams. Sketch:

(a) $y = (f(x))^2$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

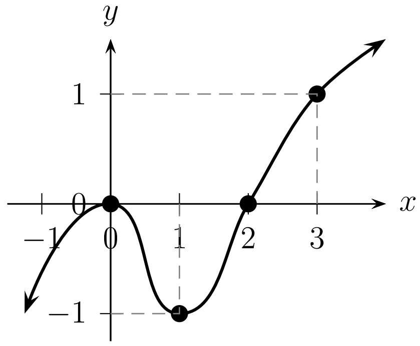
(i)



(iii)



(ii)



2. Sketch the following graphs:

(a) $y = 2 - x$ and $y = \pm\sqrt{2-x}$

(b) $y = 3x + 1$ and $y = \pm\sqrt{3x+1}$

(c) $y = x^2 - 4$ and $y^2 = x^2 - 4$

(d) $y = 9 - x^2$ and $y^2 = 9 - x^2$

(e) $y = x^2 - x - 2$ and $y = \sqrt{x^2 - x - 2}$

(f) $y = x^2\left(x^2 - 4\right)$ and $y^2 = x^2\left(x^2 - 4\right)$

(g) $y = (x-1)^2(x+1)^3$ and $y^2 = (x-1)^2(x+1)^3$

(h) $y = (x-1)(x+1)^2(x-2)^3$ and $y^2 = (x-1)(x+1)^2(x-2)^3$

(i) $y = x^4\left(x^2 - 4\right)$ and $y^2 = x^4\left(x^2 - 4\right)$

3. Sketch the following graphs:

(a) $y = (|x| - 4)^2$

(b) $y = \left(\dfrac{x-1}{x+2}\right)^2$

(c) $y = \dfrac{x^6}{(x^2+1)^6}$

(d) $y^2 = x - \dfrac{4}{x}$

(e) $y^2 = \dfrac{x-1}{x+2}$

Further exercises

Ex 5E

- Q3-4, 7, 9, 11

Part III

Inverse functions and parametric representation

Section 10

Inverse functions

Learning Goal(s)

Knowledge
Properties of inverse functions

Skills
Determining if a function has an inverse function

Understanding
The difference between inverse relations and inverse functions

By the end of this section am I able to:

3.13 Define the inverse relation of a function $y = f(x)$ to be the relation obtained by reversing all the ordered pairs of the function.

3.14 Examine and use the reflection property of the graph of a function and the graph of its inverse.

3.15 Explore reflections of functions using GeoGebra or Desmos.

3.16 Write the rule or rules for the inverse relation by exchanging x and y in the function rules, including any restrictions, and solve for y , if possible.

3.17 When the inverse relation is a function, use the notation $f^{-1}(x)$ and identify the relationships between the domains and ranges of $f(x)$ and $f^{-1}(x)$.

3.18 Apply the horizontal line test to the original function to determine if the inverse relation is also a function.

3.19 When the inverse relation is not a function, restrict the domain to obtain new functions that are one-to-one, and compare the effectiveness of different restrictions.

3.20 Solve problems based on the relationship between a function and its inverse function using algebraic or graphical techniques.

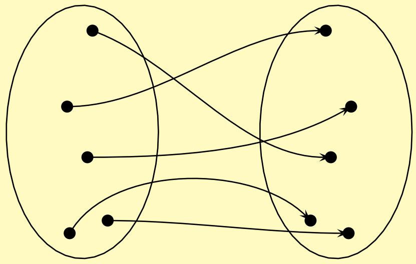
10.1 Conditions for existence of an inverse function

Definition 2

An *inverse relation* to $f(x)$ is denoted $f^{-1}(x)$. It "undo"s whatever $f(x)$ would do.

Laws/Results

A function only has an inverse if and only if it is ....



⚠ Functions/relations will always have inverse *relations*.

⚠ "Inverse" almost always implies "inverse ....", not inverse ....
.

Definition 3

Monotonic functions a .... function is also *monotonic increasing* (or *monotonic decreasing*): if $b > a$, then $f(b) > f(a)$.

Important note

⚠ Asymptotes *may* cause functions to not be monotonic. Varies from case to case!

Example 40

Which of the following functions are monotonic?

1. $y = 2x$ **2.** $y = x^2$ **3.** $y = \frac{2x}{x+2}$ **4.** $y = x - \frac{1}{x}$

Exercises

Source Fitzpatrick (1984, Ex 26(a))

1. State whether the following are one-to-one functions.

(a) $f(x) = x - 2,\ x \in \mathbb{R}$

(b) $f(x) = x^2 - 4x + 1,\ x \geq 2$

(c) $f(x) = \sqrt{4 - x^2},\ x \in [-2, 2]$

(d) $f(x) = 9 - x,\ x \in \mathbb{R}$

(e) $f(x) = \dfrac{1}{9 - x},\ x \neq 9$

(f) $f(x) = |9 - x|,\ x \in \mathbb{R}$

(g) $f(x) = 9 - x^2,\ x \in \mathbb{R}$

(h) $f(x) = x^3 - 4x,\ x \in [-2, 2]$

2. Find the largest possible domain for which the following are one-to-one (monotonic increasing) functions.

(a) $f(x) = \sqrt{4 - x^2}$

(b) $f(x) = \sqrt{x^2 - 4}$

(c) $f(x) = -\dfrac{1}{x + 2}$

(d) $f(x) = 3x - x^2$

(e) $f(x) = x^2 + 6x + 8$

3. Explain why the following functions do not have an inverse function. Suggest suitable restrictions to their domain so that the restricted function may have an inverse.

(a) $f(x) = \sqrt{a^2 - x^2},\ x \in [-a, a]$

(b) $f(x) = 4 - x^2$

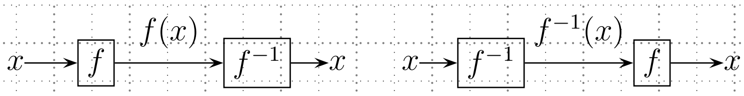
(c) $f(x) = \dfrac{1}{x^2},\ x \neq 0$

Answers

1. (a) Yes (b) Yes (c) No (d) Yes (e) Yes (f) No (g) No (h) No **2.** (a) $x \in [-2, 0]$ (b) $x \geq 2$ (c) $x < -2$ (d) $x \leq \frac{3}{2}$ (e) $x \geq -3$ **3.** These are some of the possible domain restrictions. There are others which will also work. (a) $x \in [0, a]$ (b) $x \in [0, 2]$ (c) $x \in (0, \infty)$

10.2 Properties of functions and their inverses

- Mutual inverse (over ....)



- Domain/range swap:

- Reflection about $y = x$.
  - $f(x)$ and $f^{-1}(x)$ intersect on ....
  - If f and f^{-1} intersect more than once, at least one intersection is on .... (Counterexample: ....)

- $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$ (Property for later after derivatives have been dealt with)

10.3 Finding the inverse relation

Steps

1. Interchange x and y.
2. Change subject to y.

Example 41

Find the inverse function of $y = x + 1$.

Example 42

Find the equations of the inverse relations to these functions. If the inverse is a function, find an expression for $f^{-1}(x)$, and verify that $f^{-1}\left(f(x)\right) = x$ and $f\left(f^{-1}(x)\right) = x$.

(a) $f(x) = 6 - 2x,\ x > 0.$

(b) $f(x) = x^3 + 2.$

(c) $f(x) = \dfrac{1-x}{1+x}.$

(d) $f(x) = x^2 - 9.$

Example 43

(a) Find the inverse function of $y = \dfrac{2}{x-1}$.

(b) Sketch the inverse on the same number plane as the original function.

Example 44

For the function of $f(x) = \dfrac{x+1}{x-2}$,

Answer: (a) $y = 1 + \frac{3}{x-2}$ (b) $y = 2 + \frac{3}{x+1}$

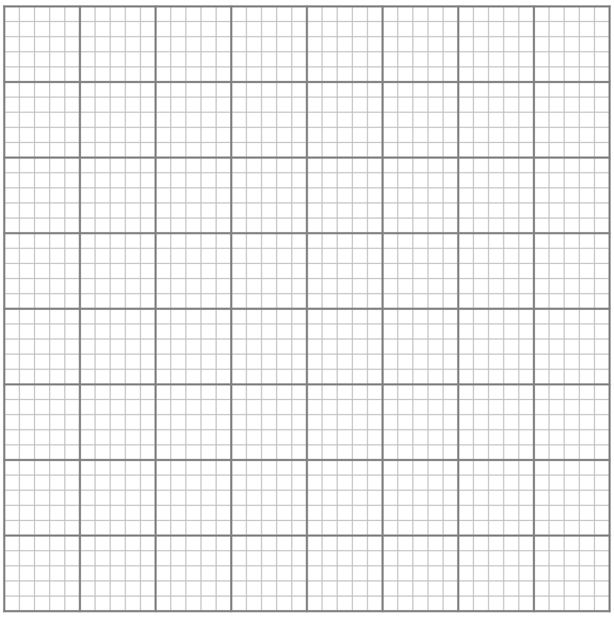
(a) Explain why $f(x)$ has an inverse function.

(b) Find the equation of the inverse function.

(c) Write down the domain and range of the inverse function.

Example 45

Find the inverse function of $y = x^3 - 2$, and sketch both curves on the same set of axes.



Example 46

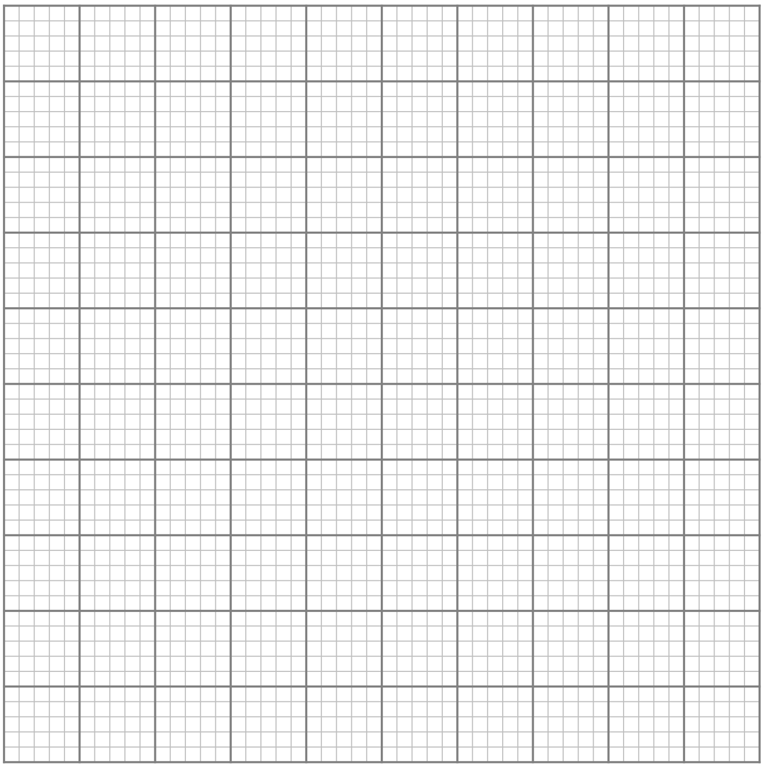
For the function $y = x^2 + 2x + 3$:

(a) State its domain and range.

(b) Sketch the curve $y = f(x)$.

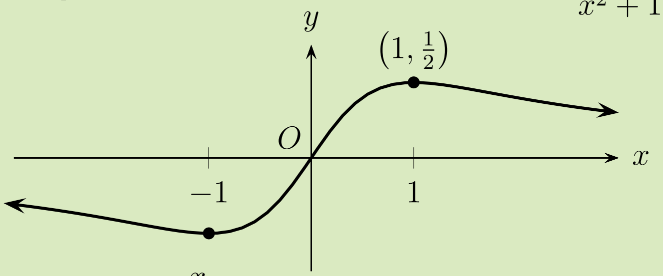
(c) Explain why the inverse function does not exist.

(d) Restrict the domain such that the curve is monotonically increasing, sketch the inverse $f^{-1}(x)$ on the same axes as $f(x)$, and find its equation.



Example 47

[2018 Ext 1 HSC Q13] The diagram shows the graph $y = \dfrac{x}{x^2+1}$, for all real x.



Consider the function $f(x) = \dfrac{x}{x^2+1}$, for $x \geq 1$.

The function $f(x)$ has an inverse. (Do NOT prove this.)

i. State the domain and range of $f^{-1}(x)$. **2**

ii. Sketch the graph $y = f^{-1}(x)$. **1**

iii. Find an expression for $f^{-1}(x)$. **3**

Example 48

[1996 3U HSC Q7] Consider the function $f(x) = \dfrac{1}{4}\left[(x-1)^2 + 7\right]$.

(i) Sketch the parabola $y = f(x)$, showing clearly any intercepts with the axes, and the coordinates of its vertex. Use the same scale on both axes. **2**

(ii) What is the largest domain containing the value $x = 3$, for which the function has an inverse function $f^{-1}(x)$? **1**

(iii) Sketch the graph of $y = f^{-1}(x)$ on the same set of axes as your graph in part (i). Label the two graphs clearly. **2**

(iv) What is the domain of the inverse function? **1**

(v) ⚠ Let a be a real number not in the domain found in part (ii). Find $f^{-1}\big(f(a)\big)$. **2**

(vi) Find the coordinates of any points of intersection of the two curves $y = f(x)$ and $y = f^{-1}(x)$. **2**

Further exercises

Ex 5F

- All questions

Ex 5G

- Q1-9

Exercises

1. For each *function* on p.28-29 of the Topic 1+2 booklet (Algebraic Techniques, Functions & Graphs), state whether or not it has an inverse function.

2. Find the inverse function of each of the following functions:

(a) $y = x + 1$

(b) $y = 3x - 2$

(c) $y = \dfrac{x+2}{3}$

(d) $y = x^3$

(e) $y = (x+1)^3$

(f) $y = \dfrac{1}{x-1}$

(g) $y = \dfrac{x}{x-1}$

3. The function $y = x$ is *invariant under inversion*, i.e. the equation of the function and its inverse are the same.

(a) Give two more examples of functions that are invariant under inversion.

(b) What do you notice about the the graphs of these functions?

4. (a) Sketch the graph of $y = \sqrt{3-x}$.

(b) On the same axes, sketch the graph of its inverse function.

(c) Find the equation of the inverse function.

(d) Find the coordinates of the point of intersection of the function and its inverse.

5. Show that the following pairs of functions are inverses by showing that over the restricted domain,

$$f \circ g(x) = g \circ f(x) = x$$

(a) $\begin{cases} f(x) = 2x - 1 \\ g(x) = \dfrac{1}{2}(x+1) \end{cases}$

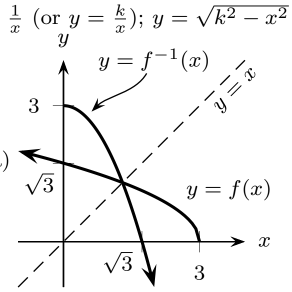
(b) $\begin{cases} f(x) = \sqrt{16 - x^2} & x \in [-4, 0] \\ g(x) = -\sqrt{16 - x^2} & x \in [0, 4] \end{cases}$

(c) $\begin{cases} f(x) = 2x - x^2 & x \geq 1 \\ g(x) = 1 + \sqrt{1-x} & x \leq 1 \end{cases}$

(d) $\begin{cases} f(x) = \dfrac{1}{2x-1} & x > \dfrac{1}{2} \\ g(x) = \dfrac{x+1}{2x} & x > 0 \end{cases}$

Answers

1. (a) Yes (b) Not function (c) No (d) Yes (e) No (f) Not function (g) Not function (h) Yes (i) Not function (j) No (k) Not function **2.** (a) $y = x - 1$ (b) $y = \frac{x+2}{3}$ (c) $y = 3x - 2$ (d) $y = \sqrt[3]{x}$ (e) $y = \sqrt[3]{x} - 1$ (f) $y = 1 + \frac{1}{x}$ (g) $y = \frac{x}{x-1}$ **3.** (a) $y = -x$ (or more generally, $y = k - x$, k being any constant); $y = \frac{1}{x}$ (or $y = \frac{k}{x}$); $y = \sqrt{k^2 - x^2}$ over $D = \{x : 0 \leq x \leq k\}$; plus an infinite number of others. (b) Symmetrical about $y = x$. **4.** (a)



(b) See previous. (c) $y = 3 - x^2$, $x \geq 0$ (d) $\left(\frac{\sqrt{13}-1}{2}, \frac{\sqrt{13}-1}{2}\right)$

Section 11

Parametric representation

Learning Goal(s)

Knowledge
Parametric form of functions and relations

Skills
Converting Cartesian form to parametric form

Understanding
The additional parameter used to represent the curve in terms of x and y values

By the end of this section am I able to:

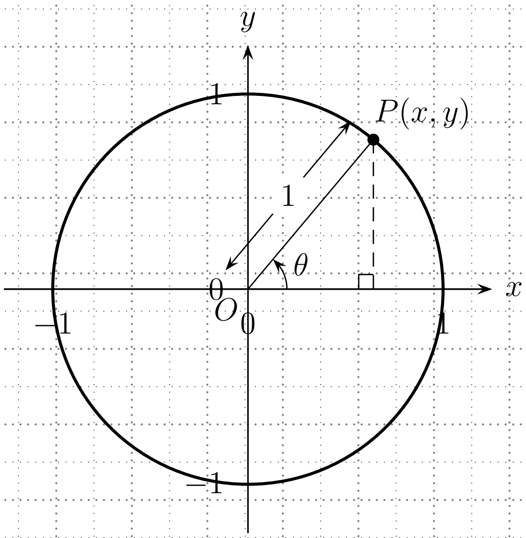
3.21 Understand the concept of parametric representation and examine lines, parabolas and circles expressed in parametric form.

3.22 The parameter t may be eliminated to give a single Cartesian equation in x and y.

11.1 Introduction

Example 49

Write down the equation of a circle with radius 1, centred at the origin in two different forms.



Definition 4

A *Cartesian* equation uses *rectangular coordinates* (x axis $\perp$ y axis). Named after René Descartes (1596–1650). `http://en.wikipedia.org/wiki/Rene_Descartes`)

- Other coordinate systems include
  - logarithmic $(x, \log_{10} x)$ – Richter scale, dB.
  - (x2) polar (r, θ) – navigation, bearings
  - cylindrical (3D polar) (ρ, θ, ϕ) – astronomy, cartography (GPS).

Definition 5

A *parametric* equation uses additional *parameter*(s) to represent the curve in terms of its x and y values.

⚠ There may be more than one parametric representation for the same curve.

11.1.1 Common parameterisations

Circle $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

- r: ....

Parabola $\begin{cases} x = 2at \\ y = at^2 \end{cases}$

- a: ....

11.1.2 Other parameterisations

- t for time in 2 dimensional motion, such as projectile motion.

$$\begin{cases} x = Vt\cos\alpha \\ y = -\frac{1}{2}gt^2 + Vt\sin\alpha \end{cases}$$

- Previously in the legacy '4 Unit' course:

Ellipse $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$ $(a \neq b,\ a, b \neq 0)$

  - a: ....
  - b: ....

Hyperbola $\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$

  - a: ....

Rectangular hyperbola $\begin{cases} x = ct \\ y = \dfrac{c}{t} \end{cases}$

11.2 Conversion from Parametric to Cartesian form

- Remove parameter by solving simultaneously

Example 50

Find the Cartesian equation of $\begin{cases} x = 3t + 1 \\ y = 2t - 3 \end{cases}$.

Answer: $2x - 3y - 11 = 0$

Example 51

Find the Cartesian equation of $\begin{cases} x = 2q \\ y = q^2 - 3 \end{cases}$.

Answer: $x^2 = 4y + 12$

Example 52

[Ex 9D Q5] Find the Cartesian equation: $\begin{cases} x = t + \dfrac{1}{t} \\ y = t^2 + \dfrac{1}{t^2} \end{cases}$

Answer: $y = x^2 - 2$, $x \leq -2$ or $x \geq 2$

Exercises

Find the Cartesian equation of the curves whose parametric equations are:

1. $\begin{cases} x = 2t \\ y = t + 2 \end{cases} \quad (t \in \mathbb{R})$

2. $\begin{cases} x = t \\ y = t^2 \end{cases} \quad (t \in \mathbb{R})$

3. $\begin{cases} x = t \\ y = \frac{1}{t} \end{cases} \quad (t \in \mathbb{R})$

4. $\begin{cases} x = t + 3 \\ y = t^2 - 5 \end{cases} \quad (t \geq 0)$

5. $\begin{cases} x = 2u - 2 \\ y = 3u + 1 \end{cases} \quad (u \in [1, 3])$

6. $\begin{cases} x = v^3 \\ y = 1 - v^2 \end{cases} \quad (v \in [-1, 1])$

7. $\begin{cases} x = t + 2 \\ y = t^2 - 1 \end{cases} \quad (t \in \mathbb{R})$

8. $\begin{cases} x = 2t^2 \\ y = 4t \end{cases} \quad (t \in \mathbb{R})$

9. $\begin{cases} x = \dfrac{2t}{1 + t^2} \\ y = \dfrac{1 - t^2}{1 + t^2} \end{cases} \quad (t \in \mathbb{R})$

Source Fitzpatrick (1984, Ex 24(b))

Answers

1. $2y = x + 4,\ x \in \mathbb{R}$ **2.** $y = x^2,\ x \in \mathbb{R}$ **3.** $y = \dfrac{1}{x},\ x \neq 0$ **4.** $x^2 - 6x + 4,\ x \in [3, \infty)$ **5.** $2y = 3x + 8,\ x \in [0, 4]$ **6.** $y = 1 - x^{\frac{2}{3}}$, $x \in [-1, 1]$ **7.** $y = x^2 - 4x + 3,\ x \in \mathbb{R}$ **8.** $y^2 = 8x,\ x \in [0, \infty)$ **9.** $x^2 + y^2 = 1,\ x \in [-1, 1]$

Further exercises

Ex 5H

- Q6-12

References

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